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The Cayley–Hamilton theorem for supermatrices

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Abstract. Starting from the expression for the superdeterminant of $(xI - M)$, where M is an arbitrary supermatrix, we propose a definition for the corresponding characteristic polynomial and we prove that each supermatrix satisfies its characteristic equation. Depending upon the factorization properties of the basic polynomials whose ratio defines the above mentioned superdeterminant we are able to construct polynomials of lower degree which are also shown to be annihilated by the supermatrix. Some particular cases and examples are discussed.

1. Introduction

Given any $n \times n$ real matrix M, its characteristic polynomial is defined by $P(x) = \det(xI - M)$, where I denotes the $n \times n$ identity matrix and x is a real variable. In general $P(x) = x^n + \sum_{k=0}^{n-1} c_k x^k$ is a monic polynomial of degree n. The Cayley–Hamilton theorem asserts that $P(x = M) = 0$. That is to say, if we substitute in $P(x)$ the real variable x by the matrix M in all the powers $x^k (k \neq 0)$, and set $x^0 = I$, we obtain the matrix zero as the result. This is a powerful theorem in the sense that it produces n^2 null identities among the matrix elements. The coefficients $c_k (k \neq 0)$ can be written in terms of $\mathrm{Tr}(M), \mathrm{Tr}(M^2), \ldots, \mathrm{Tr}(M^{n-1})$ together with their powers and $c_0 = \det(M)$. This theorem has recently found interesting applications in $(2+1)$-dimensional Chern–Simons (CS) theories [1]. Pure CS theories are of topological nature and the fundamental degrees of freedom are the traces of group elements constructed as the holonomies (or Wilson lines, or integrated connections) of the gauge connection around oriented closed curves on the manifold. The observables are the expectation values of the Wilson lines which turned out to be realized as the various knot polynomials known to mathematicians [2]. Since CS theories are also exactly soluble and possess a finite number of degrees of freedom [3], another aspect of interest is the reduction of the initially infinite-dimensional phase space to the subspace of the true degrees of freedom. The Cayley–Hamilton theorem has played an important role in the construction of the so called skein relations [4], which are relevant to the calculation of expectation values, and also in the process of reduction of the phase space. To illustrate the basic ideas related to this last point let us consider the simple case of two matrices M_1 and M_2 which belong to $SL(2, \mathbb{R})$. In this case the characteristic polynomial is $P(x) = x^2 - \mathrm{Tr}(M_1)x + 1$ and we have the Cayley–Hamilton matrix identity

$$(M_1)^2 - \mathrm{Tr}(M_1)M_1 + I = 0. \tag{1.1}$$

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By multiplying (1.1) by $M_2 M_1^{-1}$ and tracing we obtain the following relation among the traces

$$\mathrm{Tr}(M_2 M_1^{-1}) + \mathrm{Tr}(M_1 M_2) = \mathrm{Tr}(M_1)\,\mathrm{Tr}(M_2)\,. \tag{1.2}$$

We recall also that for any $SL(2,\mathbb{R})$ matrix we have $\mathrm{Tr}(M) = \mathrm{Tr}(M^{-1})$. The expression (1.2) finds a very useful application in the discussion of the reduced phase space of the de Sitter gravity in $2+1$ dimensions, which is equivalent to the Chern–Simons theory of the group $SO(2,2)$ [3]. This theory can be more easily described in terms of two copies of the group $SL(2,\mathbb{R})$, which is the spinorial group of $SO(2,2)$. The gauge invariant degrees of freedom associated to one genus of an arbitrary genus g two-dimensional surface turn out to be traces of any product of powers of two $SL(2,\mathbb{R})$ matrices M_1 and M_2, which correspond to the holonomies (or integrated connections) of the two homotopically distinct trajectories on one genus. Nevertheless, because Chern–Simons theories have a finite number of degrees of freedom, one should be able to reduce this infinite set of traces to a finite one. This task can in fact be accomplished by virtue of the relation (1.2). In other words, the trace $\mathrm{Tr}({M_1}^{p_1}{M_2}^{q_1}{M_1}^{p_2}{M_2}^{q_2}\ldots{M_1}^{p_n}{M_2}^{q_n}\ldots)$, for any p_i, q_i in $\mathcal{Z}$, can be shown to be reducible and to be expressed as a function of three traces only: $\mathrm{Tr}(M_1), \mathrm{Tr}(M_2)$ and $\mathrm{Tr}(M_1 M_2)$ [5]. A simple case of such reduction is to consider $\mathrm{Tr}(M_1^2 M_2)$ for example. Here we apply the relation (1.2) with $M_1 \to M_1$ and $M_2 \to M_1 M_2$ obtaining

$$\mathrm{Tr}(M_1^2 M_2) = \mathrm{Tr}(M_1)\,\mathrm{Tr}(M_1 M_2) - \mathrm{Tr}(M_2)\,. \tag{1.3}$$

A similar reduction can be performed in the case of $2+1$ super de Sitter gravity, which is the Chern–Simons theory of the supergroup $Osp(1|2,\mathbb{C})$ [6]. The novelty here is that one is dealing with supermatrices instead of ordinary matrices. In the particular case considered, a Cayley–Hamilton identity for the supermatrices was obtained in an heuristical way and a relation analogous to (1.2) was derived. This allowed to carry out the reduction of the infinite-dimensional phase space, this time in terms of five complex supertraces [7]. We observe that the nonlinear constraints among the traces that need to be solved in order to accomplish the reduction of the phase space, of which (1.2) is an example, are usually obtained using the so called Mandelstam identities [8]. The discussion of the relation among these two alternatives together with the construction of the latter identities in the case of supermatrices is reported in [9].

In this paper we discuss the general construction of Cayley–Hamilton type identities for supermatrices. This is an interesting problem in its own, besides the possible applications in the study of the reduced space in Chern–Simons theories defined over a supergroup. In section 2 we introduce our notation together with a number of results which will be useful for our purposes. In this section we also propose a definition of the characteristic and null polynomials for supermatrices starting from the corresponding superdeterminant. In section 3 we prove the Cayley–Hamilton theorem for the polynomials previously defined, by introducing the analogue of the adjoint for supermatrices. The main results contained in sections 2 and 3 have been already reported as a letter [10]. They are included here to make this paper self-contained and also to allow for a more detailed and precise presentation. Section 4 contains a discussion of some interesting cases together with many specific examples. Finally, in section 5 we give a short summary of this work emphasizing those points, that in our opinion, require further developement. There is also one appendix where some useful results are collected.

2. The characteristic and null polynomials for supermatrices

We consider a Grassmann algebra Λ over the complex numbers $\mathbb{C}$, following the notation stated in the appendix.

A $(p+q) \times (p+q)$ supermatrix is a block matrix of the form

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{2.1}$$

where A, B, C and D are $p \times p, p \times q, q \times p, q \times q$ matrices respectively. The distinguishing feature with respect to an ordinary matrix is that the matrix elements M_{RS}, $R = (i, \alpha)$, $S = (j, \beta)$ are elements of Λ with the property that A_{ij} $(i, j = 1, \ldots p)$ and $D_{\alpha\beta}$ $(\alpha, \beta = 1, \ldots q)$ are even elements, while $B_{i\alpha}$ and $C_{\beta j}$ are odd elements of the algebra. In particular this means that such numbers satisfy

$$B_{i\alpha} B_{j\beta} = -B_{j\beta} B_{i\alpha} \qquad C_{\alpha i} C_{\beta j} = -C_{\beta j} C_{\alpha i} \qquad B_{i\alpha} C_{\beta j} = -C_{\beta j} B_{i\alpha} \tag{2.2}$$

while A_{ij} and $D_{\alpha\beta}$ commute with everything.

Let us recall that the ordinary matrix addition and the ordinary matrix product of two supermatrices is again a supermatrix. Nevertheless, such concepts as the trace and the determinant need to be redefined, because of the odd component piece of the supermatrix.

The basic invariant under similarity transformations for supermatrices is the supertrace, defined by

$$\mathrm{Str}(M) = \mathrm{Tr}(A) - \mathrm{Tr}(D) \tag{2.3}$$

where the trace over the even matrices is the standard one. An important property of the above definition is the cyclic identity $\mathrm{Str}(M_1 M_2) = \mathrm{Str}(M_2 M_1)$, for arbitrary supermatrices, which is just a consequence of the relative minus sign in (2.3). The generalization of the determinant, called the superdeterminant, is obtained from (2.3) by defining

$$\delta \ln(S\mathrm{det} M) = \mathrm{Str}(M^{-1} \delta M) \tag{2.4}$$

with appropriate boundary conditions. In this compact notation we are summarizing the $(p+q)^2$ relations which give the partial derivatives of the function $\ln(S\mathrm{det}M)$ with respect to the entries M_{RS} of the supermatrix, in terms of the elements of the inverse supermatrix M^{-1}. For example, $\partial \ln(S\mathrm{det}M)/\partial M_{ij} = (M^{-1})_{ji}$ for the even indices i, j. These first order partial differential equations are subsequently integrated under the boundary conditions $S\mathrm{det}I = 1$, where I is the unit supermatrix, to produce the following equivalent two forms of calculating the superdeterminant [11]

$$S\mathrm{det}(M) = \frac{\det(A - BD^{-1}C)}{\det D} = \frac{\det A}{\det(D - CA^{-1}B)}. \tag{2.5}$$

All the matrices involved now are even in the Grassmann algebra and det has its usual meaning. The superdeterminant inherits the basic property $S\mathrm{det}(M_1 M_2) = S\mathrm{det}(M_2 M_1)$ and requires $\det D \neq 0$ and $\det A \neq 0$ in order to be defined. An explicit demonstration of the equality of the two alternative ways of calculating $S\mathrm{det}(M)$ is given in [12].

In order to proceed we introduce $a(x) = \det(xI - A)$ and $d(x) = \det(xI - D)$, which are the characteristic polynomials of the even matrices A and D.

Starting from the two alternatives (2.5) of calculating the superdeterminant we find it convenient to state the following:

Lemma 2.1. For any $(p+q)\times(p+q)$ supermatrix M, the characteristic function $h(x) = \mathrm{Sdet}(xI - M)$ can be written as

$$h(x) = \frac{\tilde{F}(x)}{\tilde{G}(x)} = \frac{F(x)}{G(x)} \tag{2.6}$$

where the basic polynomials $\tilde{F}$, $\tilde{G}$, F and G are given by

$$\tilde{F}(x) = \det(d(x)(xI - A) - B\,\mathrm{adj}(xI - D)C) \qquad \tilde{G}(x) = (d(x))^{p+1} \tag{2.7a}$$

$$F(x) = (a(x))^{q+1} \qquad G(x) = \det(a(x)(xI - D) - C\,\mathrm{adj}(xI - A)B)\,. \tag{2.7b}$$

Proof. The above expressions are obtained from (2.5) using the relation $(xI - Q)^{-1} = [\det(xI - Q)]^{-1}\mathrm{adj}(xI - Q)$ valid for any even matrix Q. Notice that $\tilde{F}$ is expressed in terms of the determinant of a $p \times p$ even matrix, while $G(x)$ is the determinant of a $q \times q$ even matrix.

In order to motivate the basic idea of our definition for the characteristic polynomial of a supermatrix let us consider the simple case of a block-diagonal supermatrix M (i.e. $B = 0, C = 0$). Here $h(x) = a(x)/d(x)$ and clearly the characteristic polynomial is $P(x) = a(x)d(x)$, which is the product of the numerator and the denominator of the corresponding superdeterminant. In fact we have

$$P(M) = \begin{pmatrix} a(A) & 0 \\ 0 & a(D) \end{pmatrix} \begin{pmatrix} d(A) & 0 \\ 0 & d(D) \end{pmatrix} \equiv 0 \tag{2.8}$$

because $a(A) = 0, d(D) = 0$. In the general case where $h(x)$ is given by (2.6), the numerator of the superdeterminant is $\tilde{F}$ (F) while the denominator is $\tilde{G}$ (G), which motivates the following:

Definition 2.1. For an arbitrary $(p+q)\times(p+q)$ supermatrix M we define the characteristic polynomial

$$\mathcal{P}(x) = \tilde{F}(x)G(x) = F(x)\tilde{G}(x) \tag{2.9}$$

where the basic polynomials $\tilde{F}$, $\tilde{G}$, F, and G are given in (2.7).

The fact that we choose to define $\mathcal{P}(x)$ in this manner instead of either $\mathcal{P}(x) = \tilde{F}(x)\tilde{G}(x)$ or $\mathcal{P}(x) = F(x)G(x)$ can only be justified *a posteriori*. Using (2.7) we obtain that $\mathcal{P}(x) = a(x)^{q+1}d(x)^{p+1}$. For notational simplicity we will not necessarily write explicitly the x-dependence on many of the polynomials considered in the sequel.

In the block-diagonal case where $a(x)$ and $d(x)$ have a common factor $f(x)$

$$a(x) = f(x)a_1(x) \qquad d(x) = f(x)d_1(x) \tag{2.10}$$

the characteristic polynomial is given by $P(x) = f(x)a_1(x)d_1(x)$, which is a polynomial of lower degree than the product $a(x)d(x)$. Motivated by this fact together with the work of [13], we have realized that there are some cases in which we can construct null polynomials of lower degree than $\mathcal{P}(x)$, according to the factorization properties of the basic polynomials $\tilde{F}$, $\tilde{G}$, F, G. At this point it is important to observe that we do not have a unique factorization theorem for polynomials defined over a Grassmann algebra. This can be seen, for example, from the identity $x^2 = (x+\alpha)(x-\alpha)$, where α is an even Grassmann with $\alpha^2 = 0$. The construction of such null polynomials of lower degree starts from finding the divisors of maximum degree of the pairs $\tilde{F}, \tilde{G}$, (F, G) which we denote by $R(S)$ respectively. This means that one is able to write

$$\tilde{F} = R\tilde{f} \qquad \tilde{G} = R\tilde{g} \qquad F = Sf \qquad G = Sg \tag{2.11}$$

where all polynomials are monic and also $\tilde{f}, \tilde{g}, f, g$ are of least degree by construction. They must satisfy

$$\tilde{f}/\tilde{g} = f/g \tag{2.12}$$

because of (2.6) and the expressions in (2.11) might be not unique. Let us observe that in the case of polynomials over the complex numbers (2.12) would imply at most $\tilde{f} = \lambda f, \tilde{g} = \lambda g$ with λ being a constant. Since we are considering polynomials over a Grassmann algebra this is not necessarily true as can be seen again in the above mentioned identity $x/(x-\alpha) = (x+\alpha)/x$, which we have rewritten in a convenient way. The above discussion leads us to the following:

Definition 2.2. Given an arbitrary $(p+q) \times (p+q)$ supermatrix M, with a characteristic function $h(x)$ such that $\tilde{F}, \tilde{G}$ have a common factor $R(\tilde{F} = R\tilde{f}, \tilde{G} = R\tilde{g})$ and F, G have a common factor S $(F = Sf,\ G = Sg)$, we define a null polynomial of M by

$$P(x) = \tilde{f}(x)g(x) = f(x)\tilde{g}(x)\,. \tag{2.13}$$

The above polynomial is clearly of lower degree than $\mathcal{P}(x)$, which is just a particular case of the null polynomial (2.13) when $R = S = 1$. We will concentrate mostly on definition 2.2 in the sequel.

3. The Cayley–Hamilton theorem for supermatrices

Part of our strategy to prove such a theorem for the polynomial introduced in definition 2.2 is based on one of the standard methods to prove the Cayley–Hamilton theorem for ordinary matrices [14]. We briefly recall such a procedure and emphasize that it is independent of the matrix considered being a standard matrix or a supermatrix.

Lemma 3.1. Let M, $(xI - M)$ and $N(x)$ be $(p+q) \times (p+q)$ supermatrices where M is independent of $x \in \Lambda_0$, with $N(x)$ being a polynomial supermatrix of degree $(n-1)$, $N(x) = N_0x^{n-1} + N_1x^{n-2} + \cdots + N_{n-1}x^0$, (where each $N_k (k = 0, \ldots, n-1)$ is a $(p+q) \times (p+q)$ supermatrix independent of x) such that

$$(xI - M)N(x) = P(x)I \tag{3.1}$$

where $P(x) = p_0x^n + p_1x^{n-1} + \cdots + p_nx^0$ is a numerical polynomial of degree n over Λ_0, then $P(M) = p_0M^n + \cdots + p_nI \equiv 0$.

Proof. The proof follows by comparing the independent powers of x in (3.1) and then explicitly computing $P(M)$ [14].

In the standard case the matrix $N(x)$ is just given by $N(x) = \mathrm{adj}(xI - M) = \det(xI - M)(xI - M)^{-1}$, and $P(x) = \det(xI - M)$. In the case of a supermatrix we do not have an obvious generalization either of the polynomial matrix $\mathrm{adj}(xI - M)$ or of $\det(xI - M)$. Nevertheless, following the analogy as closely as possible we define

$$N(x) = P(x)(xI - M)^{-1} \tag{3.2}$$

where $P(x)$ is the polynomial introduced in definition 2.2 of the previous section. The challenge now is to prove that $N(x)$, which trivially satisfies (3.1), is indeed a polynomial matrix. In this way we would have proved that $P(M) = 0$, according to lemma 3.1.

Lemma 3.2. Let M and $(xI - M)$ be $(p+q) \times (p+q)$ supermatrices, $x \in \Lambda_0$, then

$$(xI - M)^{-1}_{ij} = -\frac{1}{\tilde{F}}\frac{\partial \tilde{F}}{\partial A_{ji}} \qquad (xI - M)^{-1}_{i\alpha} = \frac{1}{G}\frac{\partial G}{\partial C_{\alpha i}} \tag{3.3a}$$

$$(xI - M)^{-1}_{\alpha j} = \frac{1}{\tilde{F}}\frac{\partial \tilde{F}}{\partial B_{j\alpha}} \qquad (xI - M)^{-1}_{\alpha\beta} = -\frac{1}{G}\frac{\partial G}{\partial D_{\beta\alpha}} \tag{3.3b}$$

where A_{ji}, $B_{j\alpha}$, $C_{\alpha j}$ and $D_{\alpha\beta}$ are the entries of the supermatrix M defined in (2.1) and $\tilde{F}$, G, are the polynomials given in (2.7). The derivative with respect to an odd Grassmann number is a left derivative defined such that $\delta\tilde{F} \equiv \delta B_{j\alpha}\partial\tilde{F}/\partial B_{j\alpha}$.

Proof. The first step is to calculate $(xI - M)^{-1}$ in block form, with the results

$$(xI - M)^{-1}_{11} = ((xI - A) - B(xI - D)^{-1}C)^{-1} \tag{3.4a}$$

$$(xI - M)^{-1}_{12} = -(xI - A)^{-1}B((xI - D) - C(xI - A)^{-1}B)^{-1} \tag{3.4b}$$

$$(xI - M)^{-1}_{21} = -(xI - D)^{-1}C((xI - A) - B(xI - D)^{-1}C)^{-1} \tag{3.4c}$$

$$(xI - M)^{-1}_{22} = ((xI - D) - C(xI - A)^{-1}B)^{-1} \tag{3.4d}$$

where the subindices 11, 12, 21 and 22 denote the corresponding $p \times p$, $p \times q$, $q \times p$, and $q \times q$ blocks respectively. Let us concentrate now in the 11 block. Rewritting all the inverse matrices in (3.4a) in terms of their adjoints together with the corresponding determinants we obtain

$$(xI - M)^{-1}_{11} = \frac{d}{\tilde{F}}\mathrm{adj}((xI - A)d - B\,\mathrm{adj}(xI - D)C)\,. \tag{3.5}$$

Using the basic property

$$\delta\det Q = \mathrm{Tr}(\mathrm{adj}\,Q\,\delta Q)$$

valid for any even matrix Q, we calculate the change of $\tilde{F}$ with respect to A_{ij}, keeping constant all other entries, obtaining

$$\delta\tilde{F} = -d[\mathrm{adj}((xI - A)d - B\,\mathrm{adj}(xI - D)C)]_{ij}\delta A_{ji} \tag{3.7}$$

which can be written as

$$\frac{\partial\tilde{F}}{\partial A_{ji}} = -d[\mathrm{adj}(xI - A)d - B\,\mathrm{adj}(xI - D)C)]_{ij}\,. \tag{3.8}$$

The comparison of (3.8) with (3.5) completes the proof of the first relation in (3.3a). The corresponding proof for the remaining equations (3.3) is performed following a similar procedure.

Notice that the conditions for the existence of $(xI - M)^{-1}$ are the same as those for the existence of $S\det(xI - M)$ and they are $\det(xI - A) \neq 0$ and $\det(xI - D) \neq 0$. Since x is a generic even Grassmann variable we will assume that this is always the case. By virtue of these assumptions the term $((xI - A) - B(xI - D)^{-1}C)^{-1}$, for example, can always be calculated as $(I - (xI - A)^{-1}B(xI - D)^{-1})C)^{-1}(xI - A)^{-1}$. The factor on the left can be thought of as a series expansion of the form $1/(1 - z) = 1 + z + z^2 + \cdots$, with $z = (xI - A)^{-1}B(xI - D)^{-1}C$. Moreover, the series will stop at some power because z is a matrix with body zero and thus is nilpotent.

Now we come to the principal result of this paper, which we state as the following:

Theorem 3.1. Let M and $(xI - M)$ be $(p+q) \times (p+q)$ supermatrices, $x \in \Lambda_0$, then $N(x) = P(x)(xI - M)^{-1}$, with $P(x)$ given in definition 2.2, is a polynomial matrix.

Proof. Let us consider the block-element 11 of $N(x)$ to begin with. According to lemma 3.2 together with (2.11), this block can be written as

$$N_{ij} = -g\frac{\partial \tilde{f}}{\partial A_{ji}} - \frac{g\tilde{f}}{R}\frac{\partial R}{\partial A_{ji}}. \tag{3.9}$$

The first term of the RHS is clearly of polynomial character. In order to transform the second term we make use of the property

$$\frac{\partial \ln \tilde{G}}{\partial A_{ji}} = 0 = \frac{\partial \ln R}{\partial A_{ji}} + \frac{\partial \ln \tilde{g}}{\partial A_{ji}} \tag{3.10}$$

which follows from the factorization $\tilde{G} = R\tilde{g}$, together with the fact that $\tilde{G}$ is just a function of $D_{\alpha\beta}$, according to (2.7a). In this way and also using (2.12), we obtain

$$N_{ij} = f\frac{\partial \tilde{g}}{\partial A_{ji}} - g\frac{\partial \tilde{f}}{\partial A_{ji}} \tag{3.11}$$

which leads to the conclusion that the block-matrix N_{ij} is indeed polynomial. The proof for $N_{\alpha i}$ runs along the same lines, except that now the derivatives are taken with respect to $B_{i\alpha}$ and that we have to use $\partial \ln \tilde{G}/\partial B_{i\alpha} = 0$, instead of (3.10). The remaining terms $N_{i\alpha}$ and $N_{\alpha\beta}$ can be dealt with in an analogous manner by considering the derivatives of $G = Sg$ with respect to $C_{\alpha i}$ and $D_{\beta\alpha}$, and by replacing the condition (3.10) by $\partial \ln F/\partial C_{\alpha i} = 0$ and $\partial \ln F/\partial D_{\beta\alpha} = 0$ respectively. The results are again of the form (3.11), the only difference being the variables with respect to which the derivatives are taken.

Finally, using theorem (3.1) together with lemma (3.1) we can state the following extension of the Cayley–Hamilton theorem for supermatrices:

Theorem 3.2. Let M and $(xI - M)$ be $(p+q) \times (p+q)$ supermatrices, $x \in \Lambda_0$, with $\mathrm{Sdet}(xI - M) = \tilde{F}/\tilde{G} = F/G$, where the polynomials $\tilde{F}, \tilde{G}, F$ and G are given in (2.7). Then, for any common factor R such that $\tilde{F} = R\tilde{f}, \tilde{G} = R\tilde{g}$ and S such that $F = Sf, G = Sg$, where $\tilde{f}/\tilde{g} = f/g$, the polynomial $P(x) = \tilde{f}(x)g(x) = f(x)\tilde{g}(x)$ annihilates M, i.e. $P(M) = 0$.

4. Particular cases and specific examples

In this section we consider some distinguished cases and some particular examples of null polynomials of minimum degree for supermatrices, constructed according to the definitions given in section 2. Our general procedure for constructing such null polynomials is based on the factorizaton properties of the polynomials $\tilde{F}, \tilde{G}, F$ and G introduced in section 2. The work of [13] shows that these factorization properties are closely related to those of the characteristic polynomials $a(x)$ and $d(x)$ corresponding to the even blocks of the supematrix. At this point we emphasize that when dealing with polynomials over a Grassmann algebra, the existence of a maximum common divisor of two polynomials is not in one-to-one corresponence with the fact that these polynomials are not co-prime. In fact, we will exhibit a simple example of two polynomials which are not co-prime and nevertheless do not have a common factor. In this section we will shift the emphasis to the factorization properties of $a(x)$ and $d(x)$ and we will consider three cases: (1) the polynomials a and d are co-prime, (2) the polynomials a and d are not co-prime but do not have a common factor and finally (3) both polynomials are not co-prime and have a maximum common divisor.

4.1. The polynomials $a(x)$ and $d(x)$ are co-prime

This case has been thoroughly discussed in [13] in relation with the factorization properties of $h(x)$. Theorem (3.9) of this reference proves that the characteristic function h can be written in the unique irreducible form

$$h(x) = (a+r)/(d+t) \tag{4.1}$$

where r and t are even polynomials with body zero which have the property $\deg(r) < \deg(a)$ and $\deg(t) < \deg(d)$. The two basic steps that lead to (4.1) are, in the first place, the possibility of writing

$$\tilde{F} = ad^p + u \qquad G = a^q d + v \tag{4.2}$$

together with the factorization

$$\tilde{F} = (a+r)(d^p + t') \tag{4.3a}$$

$$G = (d+t)(a^q + r') \tag{4.3b}$$

where all polynomials u, v, r, r', t, t' have body zero and $\deg(u) < p(q+1), \deg(v) < q(p+1), \deg(t') < pq, \deg(r) < p, \deg(r') < pq, \deg(t) < q$. The expressions (4.2) are just the expansions of the corresponding polynomials in (2.7) in terms of powers of the odd Grassmann numbers $B_{i\alpha}$ and $C_{\beta j}$, while the expressions (4.3a) and (4.3b) are a consequence of corollary (3.8) of [13], which we have included in the appendix for further use. The second step arises from comparing the two ways (2.6) of writing $h(x)$ and using the factorization lemma 3.4 of [13], also included in the appendix. In this way one obtains that

$$F = a^{q+1} = (a+r)(a^q + r') \tag{4.4a}$$

$$\tilde{G} = d^{p+1} = (d+t)(d^p + t')\,. \tag{4.4b}$$

The use of (4.3) and (4.4) in any expression (2.6) leads directly to the form (4.1) for $h(x)$. Our result, in the case where $a(x)$ and $d(x)$ are co-prime, is the following expression for the null polynomial of minimum degree

$$P(x) \equiv (a+r)(d+t) \quad . \tag{4.5}$$

according to theorem (3.2). Besides giving all these existence theorems, we can find in [13] what the authors call a modified Euclidean algorithm, which in fact allows us to explicitly perform the reduction in (2.6) thus obtaining the irreducible expressions appearing in (4.1).

4.1.1. (1+1) × (1+1) supermatrices. This is the simplest example of the case (4.1) and corresponds to the choice

$$M = \begin{pmatrix} p & \alpha \\ \beta & q \end{pmatrix} \tag{4.6}$$

with $\bar{p} \neq \bar{q}$ in such a way that $a = x - p$ and $d = x - q$ be co-prime polynomials, according to lemma 3.3 of [13]. The bar over a number denotes its body. Here we have

$$\begin{aligned} &\tilde{F} = (x-q)(x-p) - \alpha\beta \qquad \tilde{G} = (x-q)^2 \\ &F = (x-p)^2 \qquad G = (x-q)(x-p) + \alpha\beta\,. \end{aligned} \tag{4.7}$$

The modified Euclidean algorithm of [13] applied to each pair $\tilde{F},\ \tilde{G}$ (F, G) leads to the following factorizations

$$\begin{aligned} \tilde{F} &= \left(x - p + \frac{\alpha\beta}{q-p}\right)\left(x - q - \frac{\alpha\beta}{q-p}\right) \\ \tilde{G} &= \left(x - q + \frac{\alpha\beta}{q-p}\right)\left(x - q - \frac{\alpha\beta}{q-p}\right) \\ F &= \left(x - p + \frac{\alpha\beta}{q-p}\right)\left(x - p - \frac{\alpha\beta}{q-p}\right) \\ G &= \left(x - q + \frac{\alpha\beta}{q-p}\right)\left(x - p - \frac{\alpha\beta}{q-p}\right) \end{aligned} \tag{4.8}$$

which allow the identifications $r = -t' = t = -r' = \alpha\beta/(q-p)$, since these polynomials are of degree zero in this case. Here we have $R = (x - q - \alpha\beta/(q-p))$, $S = (x - p - \alpha\beta/(q-p))$ together with $\tilde{f} = f = x - p + \alpha\beta/(q-p)$ and $g = \tilde{g} = x - q + \alpha\beta/(q-p)$ in the notation of section 2. The null polynomial of minimum degree is then [10, 15]

$$P(x) = fg = x^2 - x\left(p + q - \frac{2\alpha\beta}{q-p}\right) + pq - \frac{(q+p)}{(q-p)}\alpha\beta \tag{4.9}$$

where we can verify by direct substitution that $P(M) \equiv 0$.

4.1.2. $Osp(1|2;\mathbb{C})$ supermatrices. Another example of this kind corresponds to the case of supermatrices belonging to the supergroup $Osp(1|2;\mathbb{C})$, which are relevant to the discussion of the reduced phase space in super de Sitter gravity in $2+1$ dimensions [6, 7]. Here we consider $(2+1) \times (2+1)$ supermatrices so that the factorization properties involved are a particular case of the example presented in [13].

The supermatrices M belonging to $Osp(1|2;\mathbb{C})$ are such that

$$M^T H M = H \qquad H = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.10}$$

where T denotes the supertransposed and H the orthosymplectic supermatrix. The above supermatrices can be parameterized in the following way:

$$M = \begin{pmatrix} A & \xi \\ \eta^t & a \end{pmatrix} \qquad \xi = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{4.11}$$

with x_1, x_2 being arbitrary odd Grassmann numbers and t denoting the standard transposition. The condition (4.10) translates into the following constraints over the remaining matrix elements

$$\eta^t = \xi^t E A \qquad a = 1 + x_1 x_2 \qquad \det A = 1 - x_1 x_2 \tag{4.12}$$

where E denotes the 2×2 antisymmetric block of H in (4.10). We assume $\mathrm{Tr}(A) \neq 2$ in order that a and d be co-prime polynomials. In the notation of [13], the unique irreducible expression for the characteristic function is

$$h = (x^2 + \sigma_1 x + \sigma_2)/(x + \sigma_3) \tag{4.13}$$

where the expressions for $\sigma_i (i = ,1, 2, 3)$ are obtained there by applying the modified Euclidean algorithm and are given in explicit form. Substituting our particular values for the supermatrix elements we obtain

$$\sigma_1 = -1 - \mathrm{Str}\, M \qquad \sigma_2 = -\sigma_3 = 1 \tag{4.14}$$

in such a way that the null polynomial of minimum degree, given by the product $(x^2 + \sigma_1 x + \sigma_2)(x + \sigma_3)$, is [7]

$$P(x) = x^3 - (2 + \mathrm{Str}\, M)(x^2 - x) - 1\,. \tag{4.15}$$

4.2. The polynomials $a(x)$ and $d(x)$ are not co-prime and do not have a common factor

An example of this kind is provided by the $(1+1) \times (1+1)$ supermatrix

$$M = \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix} \tag{4.16}$$

where σ is an even element of the Grassmann algebra such that $\tilde{\sigma} = 0$ and $\sigma^2 = 0$. In this case our procedure will produce a family of null polynomials. Here, $a = x - \sigma$ and $d = x$ which are not co-prime polynomials according to the definition of [13], because the ideal generated by a and d is not the whole ring of even polynomials over the Grassmann algebra. In particular, it is not possible to find polynomials P, Q such that $1 = Pa + Qd$. The basic reason for this is the impossibility of dividing by σ. Again, we emphasize the *unintuitive fact that even though a and d are not co-prime polynomials, they do not possess* a common factor. The basic polynomials are

$$\tilde{F} = x(x - \sigma) \qquad F = (x - \sigma)^2 = x^2 - 2x\sigma \qquad \tilde{G} = x^2 \qquad G = x(x - \sigma) \tag{4.17}$$

and we need to consider the corresponding factorization properties. It is obvious, for example, that $\tilde{F}$ and $\tilde{G}$ have x as a common factor. Surprisingly, this result can not be obtained by applying the Euclidean algorithm (or the modified Euclidean algorithm) to $\tilde{F}$ and $\tilde{G}$. The problem is that the first reminder has body zero, so that we cannot go on to the second step which requires dividing by this remainder. Besides, the non-existence of a unique factorization theorem is clearly shown here in the identity $x^2 = (x + z\sigma)(x - z\sigma)$, with z being an arbitrary complex number. Choosing $z = 1$ leads to the conclusion that $\tilde{F}$ and $\tilde{G}$ have two common factors of maximum degree which are x and $(x - \sigma)$. The same happens with F and G. Thus, after each cancellation is made, we are left with four possible combinations of the reduced ratios

$$\frac{\tilde{f}_i}{\tilde{g}_i} = \frac{f_j}{g_j} \qquad i, j = 1, 2 \tag{4.18}$$

where

$$\begin{aligned} &\tilde{f}_1 = x - \sigma \qquad \tilde{f}_2 = x \qquad f_1 = x - \sigma \qquad f_2 = x - 2\sigma \\ &\tilde{g}_1 = x \qquad \tilde{g}_2 = x + \sigma \qquad g_1 = x \qquad g_2 = x - \sigma\,. \end{aligned} \tag{4.19}$$

For each possibility one can verify that (4.18) is indeed correct. According to theorem (3.2) we obtain four null polynomials given by $P_{ij}(x) = \tilde{f}_i g_j$. They are

$$P_{11} = x^2 - \sigma x \qquad P_{12} = x^2 - 2\sigma x \qquad P_{21} = x^2 \qquad P_{22} = x^2 - \sigma x\,. \tag{4.20}$$

Since any linear combination of the above polynomials will be also annihilated by the supermatrix, we finally obtain two basic null polynomials which are

$$P_1 = x^2 \qquad P_2 = \sigma x\,. \tag{4.21}$$

4.3. The polynomials $a(x)$ and $d(x)$ are not co-prime and have a common factor

Here we consider the case where $a(x)$ and $d(x)$ have a maximum common divisor $k(x)$. That is to say we write

$$a(x) = k(x)a_1(x) \qquad d(x) = k(x)d_1(x) \tag{4.22}$$

where $a_1(x)$ and $d_1(x)$ are co-prime polynomials. We discuss the following two cases: (1) k and a_1 together with k and d_1 are co-prime polynomials and (2) k is not co-prime with a_1 and/or d_1. Each of the polynomials that we have introduced is monic. This constitutes an extension of the discussion in [13] and the next step is to consider the new factorization properties of the polynomials $\tilde{F}, \tilde{G}, F, G$.

4.3.1. The polynomial $k(x)$ is co-prime with $a_1(x)$ and $d_1(x)$. We begin by writing

$$\tilde{F} = (k^A d_1^{A-1} a_1 + Y) \qquad \tilde{G} = k^A d_1^A \qquad F = k^B a_1^B \qquad G = (k^B a_1^{B-1} d_1 + Z) \tag{4.23}$$

where Y and Z are polynomials with body zero and $A = p + 1, B = q + 1$. Using the factorization lemma 3.6 of [13] with respect to each of the prime polynomials involved we can write $\tilde{F}$ and G in the following way

$$\tilde{F} = (k^A + Y_1)(d_1^{A-1} + Y_2)(a_1 + Y_3) \qquad G = (k^B + Z_1)(a_1^{B-1} + Z_2)(d_1 + Z_3) \tag{4.24}$$

where $Y_i, Z_i, i = 1, 2, 3$ are body-zero polynomials. Using lemma 3.4 of [13] together with the fact that k, d_1, and a_1 are co-prime in pairs, we can show that the above factorization is unique. The condition $\tilde{F}G = F\tilde{G}$ and another use of lemma 3.4 leads to the following identities

$$k^{A+B} = (k^A + Y_1)(k^B + Z_1) \tag{4.25a}$$

$$d_1^A = (d_1^{A-1} + Y_2)(d_1 + Z_3) \tag{4.25b}$$

$$a_1^B = (a_1^{B-1} + Z_2)(a_1 + Y_3)\,. \tag{4.25c}$$

In this way we can identify the basic factorizations (2.11) by writing

$$\begin{aligned} &R = (d_1^{A-1} + Y_2) \qquad S = (a_1^{B-1} + Z_2) \qquad \tilde{f} = (k^A + Y_1)(a_1 + Y_3) \\ &f = k^B(a_1 + Y_3) \qquad \tilde{g} = k^A(d_1 + Z_3) \qquad g = (k^B + Z_1)(d_1 + Z_3)\,. \end{aligned} \tag{4.26}$$

Let us observe that the relation $\tilde{f}g = f\tilde{g}$ is immediately realized by virtue of (4.25*a*). Following the general procedure and modulo accidental cancellations that could occur in $(k^A + Y_1)/k^A$ or $k^B/(k^B + Z_1)$ for example, we identify

$$P(x) = k^{A+B}(a_1 + Y_3)(d_1 + Z_3) \tag{4.27}$$

as the null polynomial of minimum degree in this case.

A simple specific example of the above case is provided by the following $(2+2)\times(2+2)$ supermatrix

$$M = \begin{pmatrix} 0 & 0 & 0 & \alpha_1 \\ 0 & 1 & \alpha_2 & 0 \\ 0 & \alpha_1 & -1 & 0 \\ \alpha_2 & 0 & 0 & 0 \end{pmatrix} \tag{4.28}$$

where α_1, α_2 are odd Grassmann numbers and we define $\sigma = \alpha_1\alpha_2$, such that $\sigma^2 = \sigma\alpha_1 = \sigma\alpha_2 = 0$. Here $A = B = 3$. The basic characteristic polynomials are $a(x) = x(x-1)$ and $d(x) = x(x+1)$ so that we identify

$$k = x \qquad a_1 = x - 1 \qquad d_1 = x + 1 \tag{4.29}$$

which are indeed co-prime in pairs. The basic polynomials are

$$\begin{aligned} &\tilde{F} = x^3(x+1)^2(x-1) + \sigma x(x+1) \qquad F = x^3(x-1)^3 \\ &\tilde{G} = x^3(x+1)^3 \qquad G = x^3(x-1)^2(x+1) - \sigma x(x-1)\,. \end{aligned} \tag{4.30}$$

The induced factorization properties (4.25) are

$$\begin{aligned} &x^6 = (x^3 - \sigma x)(x^3 + \sigma x) \qquad (x+1)^3 = \left((x+1)^2 - \frac{\sigma}{2}(x+1)\right)\left(x + 1 + \frac{\sigma}{2}\right) \\ &(x-1)^3 = \left((x-1)^2 + \frac{\sigma}{2}(x-1)\right)\left(x - 1 - \frac{\sigma}{2}\right) \end{aligned} \tag{4.31}$$

from where we can read off the values for Y_i, Z_i. This, together with (4.24) and (4.30), allows to verify the factorization properties of $\tilde{F}, \tilde{G}, F, G$. In this particular example we have accidental cancellations in such a way that

$$R = x(x+1)(x + 1 - \sigma/2) \qquad S = x(x-1)(x - 1 + \sigma/2) \tag{4.32}$$

each of which differs from the corresponding expression in (4.26) by an extra factor of x. The null polynomial of minimum degree is then

$$P(x) = x^6 + \sigma x^5 - x^4 \tag{4.33}$$

which is of degree six instead of eight, due to the above mentioned accidental cancellations.

4.3.2. The polynomial $k(x)$ is not co-prime with $a_I(x)$ and/or $d_I(x)$. We can extend the previous case (4.3.1) by writing the maximum common divisor $k(x)$ as

$$k(x) = k_0 k_a k_d \tag{4.34}$$

which explicitly displays the further factorization properties involved according to the following procedure. Once k has been identified, it is written as the product of its minimun degree factors, which are subsequently rearranged according to the following convention: those having a common factor with $a_1(d_1)$ but not with $d_1(a_1)$ are called $k_a(k_d)$ respectively while all the remaining factors are included in k_0. We further demand that k_0, $k_a a_1$ and $k_d d_1$ be co-prime in pairs. The previous case corresponded to $k_a = k_d = 1$. Again, we start from the expressions (4.23) where k is substituted by (4.34) and we look for the factorization properties analogous to (4.24). Here one must be careful enough in keeping together any product of powers of k_a and a_1 or k_d and d_1, because the members of each pair are not respectively co-prime. In this way we obtain

$$\tilde{F} = (k_0^A + \bar{Y}_1)(k_d^A d_1^{A-1} + \bar{Y}_2)(k_a^A a_1 + \bar{Y}_3) \qquad G = (k_0^B + \bar{Z}_1)(k_a^B a_1^{B-1} + \bar{Z}_2)(k_d^B d_1 + \bar{Z}_3) \tag{4.35}$$

where $\bar{Y}_i, \bar{Z}_i, i = 1, 2, 3$ are body zero polynomials. Using the identity $\tilde{F}G = F\tilde{G}$ together with lemma (3.4) of [13] we extend the factorization properties (4.25) to

$$k_0^{A+B} = (k_0^A + \bar{Y}_1)(k_0^B + \bar{Z}_1), \tag{4.36a}$$

$$k_d^{A+B} d_1^A = (k_d^A d_1^{A-1} + \bar{Y}_2)(k_d^B d_1 + \bar{Z}_3) \tag{4.36b}$$

$$k_a^{A+B} a_1^B = (k_a^B a_1^{B-1} + \bar{Z}_2)(k_a^A a_1 + \bar{Y}_3)\,. \tag{4.36c}$$

This time we are not able to directly write $\tilde{F}$, $\tilde{G}$, F, G in the way prescribed by (2.11). Instead we can only arrive at the following general expressions

$$\tilde{F} = T\tilde{f}_1 \qquad \tilde{G} = T\tilde{g}_1 k_a^A / k_d^B \qquad F = U f_1 k_d^B / k_a^A \qquad G = U g_1 \qquad \tilde{f}_1 g_1 = f_1 \tilde{g}_1 \tag{4.37}$$

where

$$\begin{aligned} &T = (k_d^A d_1^{A-1} + \bar{Y}_2) \qquad && U = (k_a^B a_1^{B-1} + \bar{Z}_2) \qquad && \tilde{f}_1 = (k_0^A + \bar{Y}_1)(k_a^A a_1 + \bar{Y}_3) \\ &f_1 = k_0^B (k_a^A \tilde{a}_1 + \bar{Y}_3) \qquad && \tilde{g}_1 = k_0^A (k_d^B d_1 + \bar{Z}_3) \qquad && g_1 = (k_0^B + \bar{Z}_1)(k_d^B d_1 + \bar{Z}_3)\,. \end{aligned} \tag{4.38}$$

The equations (4.38) are the generalizations of (4.26) and we verify that $\tilde{f}_1 g_1 = f_1 \tilde{g}_1$ is satisfied in virtue of (4.36a). Two remarks are now in order: (i) the form (4.37) of writting F and $\tilde{G}$ is rather unpleasant because it does not clearly exhibit the polynomial character of these functions. Nevertheless we know that the products $T\tilde{g}_1$ and $U f_1$ can be divided by k_d^B and k_a^A respectively according to the factorization equations (4.36b) and (4.36c); (ii) the fact that we are not able to write $\tilde{F}$, $\tilde{G}$, F, G in the form of (2.11) means only that the method employed does not allow the general identification of a maximum common divisor in each case, as it happened previously. Nevertheless, the form (4.37) for the basic polynomials can still be used to construct a null polynomial according to the ideas of section 3. The definition of the null polynomial in this case is

$$P(x) = k_a k_d \tilde{f}_1 g_1 \tag{4.39}$$

and the proof that $P(M) = 0$ follows exactly the same steps as in theorem 3.2, with the only difference that the matrix $N(x) = P(x)(xI - M)^{-1}$ is constructed with the above $P(x)$. Let us remind the reader that we only need to prove that $N(x)$ is a polynomial supermatrix. Let us consider the 11 block of the supermatrix $N(x)$. Using lemma 3.2 together with (4.39) and the expression for $\tilde{F}$ in (4.37) we can write

$$N_{ij} = -k_a k_d g_1 \frac{\partial \tilde{f}_1}{\partial A_{ji}} - \frac{k_a k_d g_1 \tilde{f}_1}{T} \frac{\partial T}{\partial A_{ji}} . \tag{4.40}$$

The first term of the RHS is clearly of polynomial character. In order to transform the second term we make use of the property

$$\frac{\partial \ln \tilde{G}}{\partial A_{ji}} = 0 = \frac{\partial lnT}{\partial A_{ji}} + \frac{\partial \ln \tilde{g}_1}{\partial A_{ji}} + (p+1)\frac{\partial \ln k_a}{\partial A_{ji}} - (q+1)\frac{\partial \ln k_d}{\partial A_{ji}} \tag{4.41}$$

which follows from the factorization (4.37) of $\tilde{G}$, together with the fact that $\tilde{G}$ is just a function of $D_{\alpha\beta}$, according to (2.7a). In this way and using the last relation (4.37) we obtain

$$N_{ij} = k_a k_d f \frac{\partial \tilde{g}_1}{\partial A_{ji}} - k_a k_d g_1 \frac{\partial \tilde{f}_1}{\partial A_{ji}} + (p+1) g_1 \tilde{g}_1 k_d \frac{\partial k_a}{\partial A_{ji}} - (q+1) g_1 \tilde{f}_1 k_a \frac{\partial k_d}{\partial A_{ji}} \tag{4.42}$$

which leads to the conclusion that the block-matrix N_{ij} is indeed polynomial. The proof for the remaining blocks of $N(x)$ follows along similar arguments. The final conclusion is that $N(x)$ is indeed polynomial thus leading to $P(M) = 0$.

Finally we present an specific example of the previous case. Let us consider the $(2+2) \times (2+2)$ supermatrix

$$M = \begin{pmatrix} 1 & 0 & 0 & \alpha_1 \\ 0 & 0 & \alpha_2 & 0 \\ 0 & \alpha_1 & 0 & 0 \\ \alpha_2 & 0 & 0 & 0 \end{pmatrix} \tag{4.43}$$

with $\sigma = \alpha_1 \alpha_2$ as previously introduced. Now we have $a(x) = x(x-1)$ and $d(x) = x^2$, with $A = B = 3$. The maximum common divisor is $k = x$ and our conventions to denote the factors of k leads to

$$k_0 = k_a = 1 \qquad k_d = x \qquad a_1 = x - 1 \qquad d_1 = x . \tag{4.44}$$

The basic polynomials are

$$\tilde{F} = x^5(x-1) - \sigma x^3 \qquad F = x^3(x-1)^3 \qquad \tilde{G} = x^6 \qquad G = x^4(x-1)^2 + \sigma x^2(x-1) . \tag{4.45}$$

In this case, the factorization (4.36a) does not occur and the remaining ones are

$$x^9 = (x^5 + \sigma x^3(x+1))(x^4 - \sigma x^2(x+1)) \qquad (x-1)^3 = ((x-1)^2 + \sigma(x-1))(x-1-\sigma) \tag{4.46}$$

which correspond to (4.36*b*) and (4.36*c*) respectively. From these expressions we can read off the polynomials $\bar{Y}_2$, $\bar{Y}_3$, $\bar{Z}_2$, $\bar{Z}_3$ and verify the factorizations (4.35) for $\tilde{F}$ and G with the understanding that $(k_0^A + \bar{Y}_1)$ and $(k_0^B + \bar{Z}_1)$ should be replaced by one. Going back to (4.38) we find

$$\begin{aligned} T &= x^3(x^2 + \sigma(x+1)) \qquad & U &= (x-1)(x-1+\sigma) \\ \tilde{f}_1 &= f_1 = x - 1 - \sigma \qquad & \tilde{g}_1 &= g_1 = x^2(x^2 - \sigma(x+1)) \end{aligned} \tag{4.47}$$

and the null polynomial (4.39) is given by

$$P(x) = x^6 - x^5(1+2\sigma) + \sigma x^3 . \tag{4.48}$$

It is interesting to observe that accidental cancellations which occur in this case, that we are not able to describe in general, allow us to rewrite the expressions (4.37) exactly in the form (2.11), (2.12) with the following identifications

$$\begin{aligned} R &= x^2(x^2 + \sigma(x+1)) \qquad & S &= x^2(x-1)(x-1+\sigma) \\ \tilde{f} &= f = x(x-1-\sigma) \qquad & \tilde{g} &= g = x^2 - \sigma(x+1) . \end{aligned} \tag{4.49}$$

In this way we can find another null polynomial of degree lower than (4.48), which is given by

$$P_1(x) = fg = x^4 - (1+2\sigma)x^3 + \sigma x . \tag{4.50}$$

The simple form of the supermatrix (4.43) permits a direct verification that $P_1(M) = 0$.

5. Summary

Given an arbitrary supermatrix M and starting from $S\det(xI - M)$, which is naturally written as a ratio of polynomials, according to (2.6) and (2.7), we have introduced two types of null polynomials in definitions 2.1 and 2.2. We have also proved that each of them is annihilated by M, thus providing an extension of the Cayley–Hamilton theorem for supermatrices in theorem 3.2. At the level of some particular cases we have also extended some results of [13] by giving a constructive procedure to produce the required factorizations needed to construct what we have called null polynomials of minimum degree, for the case where $a(x)$ and $d(x)$ have a common factor.

In order to put our work in the right perspective and to suggest some possible lines of further research, we now make a few comments. We have called 'characteristic' the polynomial $\mathcal{P}(x)$ introduced in definition (2.1), because it is the one that can be directly associated with an arbitrary supermatrix, independently of the factorization properties of the numerator and denominator of $S\det(xI - M)$. Nevertheless, this polynomial carries very little information regarding the odd blocks of M and so far we have not studied to what extent it really characterizes the supermatrix. Our guess is that the null polynomials of minimum degree, given in definition 2.2 and emphasized in the examples, will be much more effective in this respect. Nevertheless, we are still lacking a completely general procedure or classification to determine when there would exist a maximum common divisor of the polynomials $\tilde{F}$ and $\tilde{G}$ (F and G) that are the building blocks of $S\det(xI - M)$. Since these minimum degree polynomials are not necessarily unique, we should also provide a criterion for selecting as many of them in order to completely characterize the supermatrix. One of the main points that should be clarified from an operational point of view when trying to answer the above questions is the relation between the property that two polynomials are or not co-prime and the property that they have or do not have common factors. In the case of polynomials over a Grassmann algebra there are three possibilities: (i) two polynomials are co-prime and they do not have a common factor, (ii) two polynomials are not co-prime and they have a maximum common divisor that can be calculated in the usual way using the Euclidean algorithm. Contrary to the standard case of polynomials over the complex numbers, where only these two alternatives are found, we have in our case a third possibility: (iii) two polynomials are not co-prime and nevertheless they do not have a common factor. It was precisely this case, when considered in our simple example (4.2) at the level of $a(x)$ and $d(x)$, which led to some unusual properties like $\tilde{F}$ and $\tilde{G}$ (F and G) having two maximum common divisors none of which could be obtained by using the Euclidean algorithm. From our point of view, it is clear that these matters require further understanding.

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Appendix

We consider a Grassmann algebra Λ over the complex numbers $\mathbb{C}$. Any element $a \in \Lambda$ is a sum of the body $\bar{a} \in \mathbb{C}$ plus the nilpotent element $s(a)$ called the soul. The ring of polynomials over this Grassmann algebra is denoted by $\Lambda_0[x]$ and consists of all polynomials

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$$

where a_k are even elements of the Grassmann algebra. The set of nilpotent elements of $\Lambda_0[x]$ is denoted by $\mathcal{Q} = s(\Lambda_0)[x]$. The Grassmann algebra Λ is generated by an infinite number of odd generators ξ^A. Nevertheless, when dealing with an specific supermatrix we consider only superfunctions of the given supermatrix elements. These elements will have an expansion in terms of the basis $\{\xi^A\}$, which is not relevant for our purposes [16].

Here we collect some basic results of [13] which we have used in this work. As far as possible we follow the notation and conventions of this reference and also we use their numeration for the respective lemmas and corollaries.

Two polynomials S and T in $\Lambda_0[x]$ are co-prime if the ideal generated by S and T is the whole ring $\Lambda_0[x]$.

Lemma 3.4. Let S, T, S_1, T_1 in $\Lambda_0[x]$ and suppose $\bar{S} = \bar{S}_1, \bar{T} = \bar{T}_1$ with S and T being co-prime. If $ST = S_1T_1$ then $S_1 = (1+R)S$ and $T_1 = (1/(1+R))T$ with $R \in \mathcal{Q}$. If moreover S and S_1 are monic, then $S = S_1$ and $T = T_1$.

Corollary 3.8. Let S and T be co-prime monic polynomials in $\Lambda_0[x]$ and let R be in $\mathcal{Q}$ such that $\deg(R) < \deg(ST)$. Then there exist R_1 and R_2 in $\mathcal{Q}$ such that $ST + R = (S + R_1)(T + R_2)$, with $\deg(R_1) < \deg(S)$ and $\deg(R_2) < \deg(T)$.

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[11] See for example De Witt B 1984 *Supermanifolds* (Cambridge: Cambridge University Press) p 33

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[16] For a discussion of this point see for example Henneaux M and Teitelboim C 1993 *Quantization of Gauge Systems* (Princeton University Press, New Jersey) ch 6